TUTORIAL NOTES FOR MATH4010

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1. INEQUALITIES

Let us discuss some important inequalities.

Our starting point is the definition of convexity .

Definition 1 (Convex function)**.** A function $f : I \to \mathbb{R}$ is said to be convex if

$$f[\lambda x + (1-\lambda)y] \leq \lambda f(x) + (1-\lambda) f(y), \tag{1.1}$$

for every $x, y \in I$ and $\lambda \in [0,1]$.

Theorem 2. *A function $f : I \to \mathbb{R}$is said to be concave if*

$$f[\lambda x + (1-\lambda)y] \geq \lambda f(x) + (1-\lambda) f(y), \tag{1.2}$$

for every $x, y \in I$ and $\lambda \in [0,1]$.

The first fundamental inequality is the so called Jensen's inequality.

Theorem 3 (Jensen's inequality)**.** *For $n \in \mathbb{N}$, Let $f : I \to \mathbb{R}$ be convex (or concave), then*

$$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k), \quad \left(or \quad f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \geq \sum_{k=1}^{n} \lambda_k f(x_k),\right) \tag{1.3}$$

for all $x_1, \cdots, x_n \in I$ and $0 \leq \lambda_k \leq 1, k = 1, \cdots, n$ such that $\sum\limits_{k=1}^{n} \lambda_k = 1$.

Proof. If $f : I \to \mathbb{R}$ is convex. We prove Jensen's inequality by induction.

For $n = 1$, the inequality holds trivially.

Suppose that the inequality holds for some $m \geq 1$ and for arbitrary $x_1, \cdots, x_{m+1} \in I$, $0 \leq \lambda_k \leq 1$, $k = 1, \cdots, m$, $0 < \lambda_{m+1} < 1$ and $\sum\limits_{k=1}^{m+1} \lambda_k = 1$. Let us denote $\lambda = \sum\limits_{k=1}^{m} \lambda_k$, then by the induction hypothesis,

$$\begin{aligned} f\left(\sum_{k=1}^{m+1} \lambda_k x_k\right) &= f\left(\lambda \sum_{k=1}^{m} \frac{\lambda_k}{\lambda} x_k + (1-\lambda) x_{m+1}\right) \\ &\leq \lambda f\left(\sum_{k=1}^{m} \frac{\lambda_k}{\lambda} x_k\right) + (1-\lambda) f(x_{m+1}) \\ &\leq \lambda \sum_{k=1}^{m} \frac{\lambda_k}{\lambda} f\left(x_k\right) + (1-\lambda) f(x_{m+1}) \\ &\leq \sum_{k=1}^{m+1} \lambda_k f(x_k). \end{aligned}$$

The result follows by induction.

If $f : I \to \mathbb{R}$ is concave, then $g = -f$ is concave, therefore by the above result, we have

$$g\left(\sum_{k=1}^{m+1} \lambda_k x_k\right) \leq \sum_{k=1}^{m+1} \lambda_k g(x_k),$$

therefore

$$f\left(\sum_{k=1}^{m+1} \lambda_k x_k\right) \geq \sum_{k=1}^{m+1} \lambda_k f(x_k).$$

□

With the Jensen's inequality, another useful inequality can be derived by considering the concave function $f(x) = \ln x$.

Theorem 4 (Young's inequality)**.** *Let $p, q \in \mathbb{R}$ be two conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}, \tag{1.4}$$

for all $x, y \in \mathbb{C}$.

Proof. It suffices to prove for the case that both x and y are non-zero. Since $f(x) = \ln x$ is concave, therefore by Jensen's inequality, for two conjugate exponents p and q,

$$\ln\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right) \geq \frac{1}{p}\ln(|x|^p) + \frac{1}{q}\ln(|y|^q) = \ln|x| + \ln|y|,$$

which implies the Young's inequality □

The famous Hölder's inequality can be proved by using Young's inequality.

Theorem 5 (Hölder's inequality)**.** *Let $n \in \mathbb{N}$and $p, q \in \mathbb{R}$ be two conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}, \tag{1.5}$$

for all $x_k, y_k \in \mathbb{C}, k = 1, \cdots, n$.

Proof. It suffices to prove for the case where at least one of the x_k and one of the y_k are non-zero. Let us denote

$$u = \frac{x_k}{\left(\sum\limits_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}}, \quad v = \frac{y_k}{\left(\sum\limits_{j=1}^{n} |y_j|^q\right)^{\frac{1}{q}}},$$

then by Young's inequality, for two conjugate exponents p and q,

$$\frac{|x_k y_k|}{\left(\sum\limits_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum\limits_{j=1}^{n} |y_j|^q\right)^{\frac{1}{q}}} \leq \frac{1}{p}\frac{|x_k|^p}{\sum\limits_{j=1}^{n} |x_j|^p} + \frac{1}{q}\frac{|y_k|^q}{\sum\limits_{j=1}^{n} |y_j|^q},$$

summing both sides over $k = 1, \cdots, n$,

$$\frac{\sum\limits_{k=1}^{n} |x_k y_k|}{\left(\sum\limits_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum\limits_{j=1}^{n} |y_j|^q\right)^{\frac{1}{q}}} \leq 1,$$

which implies the Hölder's inequality. □

Next we introduce another famous inequality.

Theorem 6 (Minkowski's inequality). *Let $n \in \mathbb{N}$ and $p \geq 1$, then*

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}}, \tag{1.6}$$

for all $x_k, y_k \in \mathbb{C}, k = 1, \cdots, n$.

Proof. It suffices to prove for the case where at least one of the x_k and one of the y_k are non-zero. Let us denote

$$u = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}, \quad v = \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}},$$

then by the triangle inequality,

$$|x_k + y_k|^p \leq (|x_k| + |y_k|)^p = (u+v)^p \left(\frac{u}{u+v}\frac{|x_k|}{u} + \frac{v}{u+v}\frac{|y_k|}{v}\right)^p.$$

Since $f(x) = x^p$ is convex for $p \geq 1$, by Jensen's inequality,

$$\left(\frac{u}{u+v}\frac{|x_k|}{u} + \frac{v}{u+v}\frac{|y_k|}{v}\right)^p \leq \frac{u}{u+v}\frac{|x_k|^p}{u^p} + \frac{v}{u+v}\frac{|y_k|^p}{v^p},$$

therefore

$$|x_k + y_k|^p \leq (u+v)^p \left(\frac{u}{u+v}\frac{|x_k|^p}{u^p} + \frac{v}{u+v}\frac{|y_k|^p}{v^p}\right),$$

summing both sides over $k = 1, \cdots, n$,

$$\sum_{k=1}^{n} |x_k + y_k|^p \leq \left[\left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}}\right]^p,$$

which implies the Minkowski's inequality. □

To study the various properties of functional spaces, it is necessary to extend the above fundamental inequalities to the infinite series case. More precisely, we have the following results.

Theorem 7 (Hölder's inequality). *Let $p, q \in \mathbb{R}$ be two conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. If the series $\sum\limits_{k=1}^{\infty} |x_k|^p$ and $\sum\limits_{k=1}^{\infty} |y_k|^q$ are convergent, then the series $\sum\limits_{k=1}^{\infty} |x_k y_k|$ is convergent, moreover,*

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |y_k|^q\right)^{\frac{1}{q}}, \tag{1.7}$$

for all $x_k, y_k \in \mathbb{C}, k = 1, \cdots, n$.

Proof. For arbitrary $n \in \mathbb{N}$, by Hölder's inequality,

$$\sum_{k=1}^{n} |x_k y_k| \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}} \le \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |y_k|^q\right)^{\frac{1}{q}},$$

then by letting n goes to infinity, (1.7) is valid. Therefore by the monotone convergence theorem, $\sum_{k=1}^{\infty} |x_k y_k|$ is convergent. □

Theorem 8 (Minkowski's inequality). *Let* $p \ge 1$. *If the series* $\sum_{k=1}^{\infty} |x_k|^p$ *and* $\sum_{k=1}^{\infty} |y_k|^p$ *are convergent, then the series* $\sum_{k=1}^{\infty} |x_k + y_k|^p$ *is convergent, moreover, then*

$$\left(\sum_{k=1}^{\infty} |x_k + y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{\frac{1}{p}}, \tag{1.8}$$

for all $x_k, y_k \in \mathbb{C}, k = 1, \cdots, n$.

Proof. For arbitrary $n \in \mathbb{N}$, by Minkowski's inequality,

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{\frac{1}{p}},$$

then by letting n goes to infinity, (1.8) is valid. Therefore by the monotone convergence theorem, $\sum_{k=1}^{\infty} |x_k + y_k|^p$ is convergent. □

In the following, we give some examples as applications of the above inequalities.

Example 9. For $1 \le p < \infty$, then ℓ_p, i.e.

$$\ell_p := \{(x(i)) : x(i) \in \mathbb{R}, \quad \sum_{i=1}^{\infty} |x(i)|^p < \infty\}.$$

endowed with the norm $\|x\|_p := \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{\frac{1}{p}}$ for $x \in \ell_p$ is a Banach space.

Proof. Firstly, we prove ℓ_p is a normed space. It is obvious to see $\|\cdot\|_p$ satisfies the following two properties,

(1) $\|x\|_p \ge 0$ for all $x \in \ell_p$ where the equality holds if and only if $x = 0$.
(2) $\|\alpha x\|_p = |\alpha| \cdot \|x\|_p$ for all $x \in \ell_p$ and $\alpha \in \mathbb{R}$.

Moreover, by the Minkowski's inequality, for x, y and $z \in \ell_p$,

$$\begin{aligned}\|x-y\|_p &= \left(\sum_{i=1}^{\infty} |x(i)-y(i)|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{i=1}^{\infty} |x(i)-z(i)+z(i)-y(i)|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{i=1}^{\infty} |x(i)-z(i)|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |z(i)-y(i)|^p\right)^{\frac{1}{p}} \\ &\le \|x-z\|_p + \|z-y\|_p.\end{aligned}$$

therefore $\|\cdot\|_p$ is norm on ℓ_p.

Then we prove ℓ_p is complete under the norm $\|\cdot\|_p$. Let $\{x_n\}_{n\ge1} \in \ell_p$ be a Cauchy sequence, therefore for arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$,

$$\|x_n - x_m\|_p = \left(\sum_{i=1}^{\infty} |x_n(i)-x_m(i)|^p\right)^{\frac{1}{p}} < \varepsilon,$$

hence for arbitrary $i \in \mathbb{N}$,

$$|x_n(i) - x_m(i)| < \varepsilon,$$

which implies $\{x_n(i)\}_{n\ge1}$ is a Cauchy sequence in $\mathbb{R}$, therefore since $\mathbb{R}$ is complete, there exists $x(i) \in \mathbb{R}$ such that $\{x_n(i)\}_{n\ge1}$ converges to $x(i)$, i.e. for arbitrary $i \in \mathbb{N}$ and $\varepsilon > 0$, there exists $M_i \in \mathbb{N}$ such that for all $m > M_i$,

$$|x_m(i) - x(i)| < \frac{\varepsilon}{2^{\frac{i}{p}}}.$$

Then we define $x : i \to x(i)$ and show that $x \in \ell_p$ is the limit of $\{x_n\}_{n\ge1}$ in ℓ_p.

Since by Minkowski's inequality and choosing $n > \max\{M_1, \cdots, M_k\}$,

$$\sum_{i=1}^{k} |x(i)|^p \le \sum_{i=1}^{k} |x_n(i)|^p + \sum_{i=1}^{k} |x_n(i)-x(i)|^p < \|x_n\|^p + \varepsilon^p,$$

therefore by letting k goes to infinity,

$$\sum_{i=1}^{\infty} |x(i)|^p < \infty,$$

which implies $x \in \ell_p$. Moreover, for arbitrary $k \in \mathbb{N}$ and $\varepsilon > 0$, we choose $n > N$ and $m > \max\{N, M_1, \cdots, M_k\}$,

$$\begin{aligned}\sum_{i=1}^{k} |x_n(i)-x(i)|^p &\le \sum_{i=1}^{k} |x_n(i)-x_m(i)|^p + \sum_{i=1}^{k} |x_m(i)-x(i)|^p \\ &\le \sum_{i=1}^{\infty} |x_n(i)-x_m(i)|^p + \varepsilon^p \\ &< 2\varepsilon^p,\end{aligned}$$

therefore by letting k goes to infinity

$$\sum_{i=1}^{\infty} |x_n(i) - x(i)|^p < 2\varepsilon^p,$$

which implies $\{x_n\}_{n\geq 1}$ converges to x in ℓ^p. $\square$

Example 10.

$$\ell_\infty := \{(x(i)) : x(i) \in \mathbb{R}, \sup_i |x(i)| < \infty\}$$

endowed with the norm $\|x\|_\infty := \sup_i |x(i)|$ for $x \in \ell_\infty$ is a Banach space.

Proof. Firstly, we prove ℓ_∞ is a normed space. It is obvious to see $\|\cdot\|_\infty$ satisfies the following two properties,

(1) $\|x\|_\infty \geq 0$ for all $x \in \ell_\infty$ where the equality holds if and only if $x = 0$.
(2) $\|\alpha x\|_\infty = |\alpha| \cdot \|x\|_\infty$ for all $x \in \ell_\infty$ and $\alpha \in \mathbb{R}$.

Moreover, by the triangle inequality, for x, y and $z \in \ell_\infty$,

$$\begin{aligned} \|x - y\|_\infty &= \sup_i |x(i) - y(i)| \\ &\leq \sup_i |x(i) - z(i)| + \sup_i |z(i) - y(i)| \\ &\leq \|x - z\|_\infty + \|z - y\|_\infty. \end{aligned}$$

Therefore $\|\cdot\|_\infty$ is a norm on ℓ_∞.

Then we prove ℓ_∞ is complete under the norm $\|\cdot\|_\infty$. Let $\{x_n\}_{n\geq 1} \in \ell_\infty$ be a Cauchy sequence, therefore for arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$,

$$\|x_n - x_m\|_\infty = \sup_i |x_n(i) - x_m(i)| < \varepsilon,$$

hence for every $i \in \mathbb{N}$,

$$|x_m(i) - x_n(i)| < \varepsilon,$$

which implies that $\{x_n(i)\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, therefore since $\mathbb{R}$ is complete, there exists $x(i) \in \mathbb{R}$ such that $\{x_n(i)\}_{n\geq 1}$ converges to $x(i)$, i.e. for arbitrary $i \in \mathbb{N}$ and $\varepsilon > 0$, there exists $M_i \in \mathbb{N}$ such that for all $m > M_i$,

$$|x_m(i) - x(i)| < \varepsilon.$$

Then we define $x : i \to x(i)$ and show that $x \in \ell_\infty$ is the limit of $\{x_n\}_{n\geq 1}$ in ℓ_∞.

Since by Minkowski's inequality and choosing $n > N$,

$$\sup_i |x(i)| \leq \sup_i |x_n(i)| + \sup_i |x_n(i) - x(i)| < \|x_n\|_\infty + \varepsilon,$$

which implies $x \in \ell_p$. Moreover, for arbitrary $i \in \mathbb{N}$ and $\varepsilon > 0$, we choose $n > N$ and $m > \max\{N, M_i\}$,

$$|x_n(i) - x(i)| \leq |x_n(i) - x_m(i)| + |x_m(i) - x(i)| < 2\varepsilon,$$

which implies $\{x_n\}_{n\geq 1}$ converges to x in ℓ_∞. $\square$

Example 11.

$$c_0 := \{(x(i)) : x(i) \in \mathbb{R}, \lim_{i\to\infty} |x(i)| = 0\}$$

endowed with the norm $\|\cdot\|_\infty$ is a Banach space.

Proof. Since c_0 is a subset of ℓ_∞, it suffices to prove that c_0 is closed in ℓ_∞.

Let $\{x_n\}_{n\geq 1} \subset c_0$ converges to $x \in \ell_\infty$, then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N_1$,

$$\|x_n - x\|_\infty < \frac{\varepsilon}{2},$$

hence for all $i \in \mathbb{N}$,

$$|x_n(i) - x(i)| < \frac{\varepsilon}{2},$$

because $\{x_n(i)\}_{i\geq 1}$ converges to 0 as $i \to \infty$, then there exists $N_2 \in \mathbb{N}$ such that for all $i > N_2$,

$$|x_n(i)| < \frac{\varepsilon}{2},$$

thus

$$|x(i)| \leq |x_n(i)| + |x_n(i) - x(i)| < \varepsilon,$$

which implies $x \in c_0$. Therefore c_0 is closed in ℓ_∞. □

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